# Electronic Supplementary Material

for

# Dynamics of spherical space debris of different sizes falling to Earth

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Supplementary Video Clips VC1, VC2, VC3, VC4, VC5, VC6, VC7 with Legends

Supplementary Mathematical Formulae: Gravitational Potential of the Geoid

# Supplementary Video Clips VC1, VC2, VC3, VC4, VC5, VC6, VC7 with Legends

**Video Clip VC1**: Positions (colored dots) of 1000 spherical iron (with density of $7.9 \cdot 10^3$ kg/m$^3$) particles with different radius $r_{debris}$ until 2 h 18 min 14 sec after their launch with tangential ($\alpha = 90^o$) circular orbit velocity $v_0 = 7.847$ km/s from the same point at height $h = 100$ km from the Earth's surface (see Figure 4A). The region above the Earth's surface ($h \geq 0$) is 100 times magnified for illustrative purposes. Different radii $r_{debris}$ are coded with different colors from violet for the smallest (0.01 mm) to red for the largest ($10^4$ mm) particle.

**Video Clip VC2**: Trajectories (series of dots of a given color) of 100 iron particles with different radius $r_{debris}$ until 6 h 19 min 17 sec after their launch with tangential ($\alpha = 90^o$) circular orbit velocity $v_0 = 7.817$ km/s from height $h = 150$ km (see Figure 4B). The region above the Earth's surface is 66 times magnified. Different radii $r_{debris}$ are coded with different colors from violet for the smallest (0.01 mm) to red for the largest ($10^4$ mm) particle.

**Video Clip VC3**: Trajectories of 20 iron particles with different radius $r_{debris}$ until 2 h 10 min 27 sec after their launch with tangential ($\alpha = 90^o$) circular orbit velocity $v_0 = 7.877$ km/s from height $h = 50$ km (see Figure 4C). The region above the Earth's surface is 200 times magnified. Different radii $r_{debris}$ are coded with different colors from violet for the smallest (0.01 mm) to red for the largest ($10^4$ mm) particle.

**Video Clip VC4**: Trajectories of 100 iron particles with different radius $r_{debris}$ until 2 h 18 min 14 sec after their launch with tangential ($\alpha = 90^o$) circular orbit velocity $v_0 = 7.847$ km/s from height $h = 100$ km (see Figure 4D). The region above the Earth's surface is 100 times magnified. Different radii $r_{debris}$ are coded with different colors from violet for the smallest (0.01 mm) to red for the largest ($10^4$ mm) particle.

**Video Clip VC5**: Trajectories of 100 iron particles with different radius $r_{debris}$ until 6 h 44 min 38 sec after their launch with initial velocity $v_0 = 4.57$ km/s and angle $\alpha = 122.6^o$ relative to the radial direction from the same point at height $h = 10000$ km above the Earth's surface (see Figure 5A). The region above the Earth's surface is not magnified. Different radii $r_{debris}$ are coded with different colors from violet for the smallest (0.01 mm) to red for the largest ($10^4$ mm) particle.

**Video Clip VC6**: Trajectories of 100 iron particles with different radius $r_{debris}$ until 2 h 31 min 29 sec after their launch with initial velocity $v_0 = 5.65$ km/s and angle $\alpha = 45^o$ from height $h = 100$ km (see Figure 5B). The region above the Earth's surface is 100 times magnified. Different radii $r_{debris}$ are coded with different colors from violet for the smallest (0.01 mm) to red for the largest ($10^4$ mm) particle.

**Video Clip VC7**: Trajectories of 30 (different angles $\alpha$) $\times$ 11 (different radii $r$) = 330 spherical iron particles until 1 h 54 min 20 sec after the explosion of a space debris at height $h = 1000$ km above the Earth's surface (see Figure 7). The initial velocity of all explosion fragments was $v_0 = 7$ km/s and the angle of the initial velocity vector changed from $\alpha = 0^o$ to $\alpha = 360^o$ with an increment $\Delta\alpha = 12^o$ (see Supplementary Video Clip VC7) where the region above the Earth's surface is not magnified. Different radii $r_{debris}$ are coded with different colors from violet for the smallest (0.01 mm) to red for the largest ($10^4$ mm) particle.

# Supplementary Mathematical Formulae

## Gravitational Potential of the Geoid

The gravitational potential $U$ of the Earth is expressed in the form of the following series (Klinkrad 2006d):

$$U = \frac{\gamma m_{\mathrm{E}}}{r}\left\{1 - \sum_{\mathrm{k}=2}^{\infty} J_{\mathrm{k}}\left(\frac{r_0}{r}\right)^{\mathrm{k}} P_{\mathrm{k}}(\sin\varphi) + \sum_{\mathrm{k}=2}^{\infty}\sum_{\mathrm{m}=1}^{\mathrm{k}}\left(\frac{r_0}{r}\right)^{\mathrm{k}} P_{\mathrm{k}}^{\mathrm{m}}(\sin\varphi)\left[c_{\mathrm{km}}\cos(m\lambda) + s_{\mathrm{km}}\sin(m\lambda)\right]\right\}, \quad \text{(A1)}$$

$$J_{\mathrm{k}} = -c_{\mathrm{k}0}. \quad \text{(A2)}$$

where φ and λ are the latitude and longitude of the Earth's spherical coordinate system, $\gamma = 6.67408\cdot10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$ is the universal gravitational constant, $m_{\mathrm{E}} = 5.972\cdot10^{24}$ kg is the Earth’s mass, $P_{\mathrm{k}}(\sin\varphi)$ are Legendre polynomials of degree $k$, $P_{\mathrm{k}}^{\mathrm{m}}(\sin\varphi)$ are associated Legendre functions of degree $k$ and order $m$, $r = \sqrt{x^2 + y^2}$, $r_0 = 6378$ km is the Earth’s average equatorial radius.

In (A1), the first component $\gamma m_{\mathrm{E}}/r$ is the mass point potential, which corresponds to an Earth with a spherical symmetric mass distribution. The first sum (being independent of λ) is the gravitational potential of a rotationally symmetric body. This sum includes the $P_{\mathrm{k}}(\sin\varphi)$ Legendre polynomials, which are zonal spherical harmonic functions, therefore the $J_{\mathrm{k}}$ coefficients are called zonal harmonic coefficients. The even and the odd index coefficients $k$ express the symmetry and the asymmetry of Earth to the Equator.

In the second sum of (A1), the components corresponding to $m = k$ are the so-called sectorial harmonics, so $c_{\mathrm{kk}}$ and $s_{\mathrm{kk}}$ are the sectorial harmonic coefficients. If $0 < m < k$, then $c_{\mathrm{km}}$ and $s_{\mathrm{km}}$ are the so-called tesseral harmonic coefficients. The sectorial and tesseral components express the deviations of the Earth's potential from the potential of a rotationally symmetric body. Using series (A1) and (A2), the geoid potential can be written in the following simpler form:

$$U = \frac{\gamma m_{\mathrm{E}}}{r}\{1 + Q\}, \quad \text{(A3)}$$

where

$$Q = \sum_{\mathrm{k}=2}^{\infty}\sum_{\mathrm{m}=0}^{\mathrm{k}}\left(\frac{r_0}{r}\right)^{\mathrm{k}} P_{\mathrm{k}}^{\mathrm{m}}(\sin\varphi)\left[c_{\mathrm{km}}\cos(m\lambda) + s_{\mathrm{km}}\sin(m\lambda)\right]. \quad \text{(A4)}$$

is the so-called perturbation function. The dimensionless $J_{\mathrm{k}}$ coefficients are available from tables. If $k > 2$, the values of $J_{\mathrm{k}}$ ($< 10^{-6}$) are three orders of magnitude smaller than $J_2$ ($\approx 10^{-3}$). If $m > 0$, then $c_{\mathrm{km}}$ and $s_{\mathrm{km}} < 10^{-6}$. Since we compute only the impact times and therefore we do not need high accuracy, all spherical harmonics can be neglected, except for the case $k = 2$, $m = 0$, thus we obtain:

$$P_2^0(\sin\varphi) = \frac{3}{2}\sin^2\varphi - \frac{1}{2}, \quad \text{where in our 2D model:} \quad \sin\varphi = \frac{y}{\sqrt{x^2 + y^2}}. \quad \text{(A5)}$$

Replacing (A5) into (A4), the perturbation function is:

$$Q = \left(\frac{r_0}{r}\right)^2 \left[c_{20} \cdot \cos(0) + s_{20} \cdot \sin(0)\right] P_2^0(\sin\varphi) = \left(\frac{r_0}{r}\right)^2 c_{20} P_2^0(\sin\varphi) = -\left(\frac{r_0}{r}\right)^2 J_2 P_2^0(\sin\varphi) . \qquad \text{(A6)}$$

Using (A5) and (A6), finally the perturbation function is:

$$Q = \frac{-J_2 r_0^2}{2\left(x^2+y^2\right)}\left(\frac{3y^2}{x^2+y^2}-1\right), \quad \text{where} \quad J_2 = 1.08252 \cdot 10^{-3}. \qquad \text{(A7)}$$

If we want to take into consideration the geoid shape of the Earth, using the perturbation function (A7), the following two partial derivatives should be added to the right sides of the equations of motion (2.10):

$$\frac{\partial Q}{\partial x} = \frac{-J_2 r_0^2}{\left(x^2+y^2\right)^2}\left[x - \frac{6xy^2}{x^2+y^2}\right],$$

$$\frac{\partial Q}{\partial y} = \frac{-J_2 r_0^2}{\left(x^2+y^2\right)^2}\left[y - \frac{6y^3 - 3y\left(x^2+y^2\right)}{x^2+y^2}\right]. \qquad \text{(A8)}$$