Looking into the water with oblique head tilting: revision of the aerial binocular imaging of underwater objects

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Received September 13, 2002; revised manuscript received January 23, 2003; accepted January 30, 2003

It is a well-known phenomenon that when we look into the water with two aerial eyes, both the apparent position and the apparent shape of underwater objects are different from the real ones because of refraction at the water surface. Earlier studies of the refraction-distorted structure of the underwater binocular visual field of aerial observers were restricted to either vertically or horizontally oriented eyes. We investigate a generalized version of this problem: We calculate the position of the binocular image point of an underwater object point viewed by two arbitrarily positioned aerial eyes, including oblique orientations of the eyes relative to the flat water surface. Assuming that binocular image fusion is performed by appropriate vergent eye movements to bring the object's image onto the foveas, the structure of the underwater binocular visual field is computed and visualized in different ways as a function of the relative positions of the eyes. We show that a revision of certain earlier treatments of the aerial imaging of underwater objects is necessary. We analyze and correct some widespread erroneous or incomplete representations of this classical geometric optical problem that occur in different textbooks. Improving the theory of aerial binocular imaging of underwater objects, we demonstrate that the structure of the underwater binocular visual field of aerial observers distorted by refraction is more complex than has been thought previously.

OCIS codes: 000.2690, 080.1510, 080.2720, 170.1420, 330.1400, 330.7310.

1. INTRODUCTION

Both the apparent position and the apparent shape of an underwater object viewed by two eyes from air do not coincide with the object's true position and shape because of refraction of light at the water surface. This can be experienced in everyday life when we look into the aquarium or gaze at the underwater world during fishing or boating, for example. The geometric optics of this well-known phenomenon seems to be simple at first sight, and indeed, it is usually treated in one of the introductory sections of physics textbooks. It occasionally appears also in the popular literature,[1] especially in magazines dealing with angling and fishing.[2] In the scientific literature,[3–13] two possible apparent image positions of an underwater object point O viewed from air occur (Fig. 1): the first, C, is positioned where the line of the refracted ray entering the eye touches the evolute of refracted rays, and the second, V, is at the point where the vertical line passing through O crosses the refracted ray extrapolated backward.

Some common representations of this classical geometric optical problem are shown in Figs. 2 and 3. As we show in this paper, the majority of these representations are incomplete or incorrect, which demonstrates that this problem is more complicated than it appears at a glance. The first reason for complexity is that the imaging of an underwater object point O viewed from air by a single eye lens is astigmatic owing to refraction and the nonzero diameter of the pupil.[5,13] In reality, the image of O is a slightly elongated vertical line at V (Fig. 1) with gradual blurring toward the ends, and the length of this image line depends on the shape of the pupil as well as on the distance of the eye from O. A single aerial eye must focus onto V to see O as sharply as possible.[13]

The second reason for complexity is of physiological and psychophysical origin. With one eye alone, the human visual system is unable to determine the position of an object point O in an unknown optical environment.[14,15] In principle, one of the measures of the distance of O from the eye could be the lens accommodation being in close connection with the tension of the lens muscles. A similar mechanism plays a role in the depth perception of certain animals.[16,17] In the human visual system, however, the coding of distance is not based exclusively on lens accommodation; additional information is needed, such as retinal disparities and angles of convergence of the optical axes of the eyes (binocular stereopsis). Furthermore, occlusions, angular sizes, and retinal positions of objects as well as texture gradients and motion parallax play an important role in distance estimation.[15] Thus all drawings, explanations, and derivations of the apparent image position of underwater objects viewed from air by one human

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eye alone are physiologically and psychophysically incorrect, because a single aerial human eye can perceive only the refraction-induced apparent change of the angle of elevation but not the distance of an underwater object.

The third reason for complexity is that the apparent position of an underwater object viewed binocularly from air depends on the choice of refracted rays included in image formation. Although this was recognized by, e.g., Kinsler,[4] Kedves,[5] and Buchholz,[9] they treated the binocular image formation of underwater objects only for the most usual arrangement of the eyes, namely, for eyes positioned horizontally. Horváth and Varjú[13] emphasized the strong dependence of the binocular image position of an underwater object on the relative loci of the eyes: If rays in a vertical plane, containing the underwater object point O and both eyes E1 and E2, are considered (Fig. 4), the image is C, because lines $e1$ and $e2$ of the refracted rays entering the pupils intersect at C. (To simplify indexing, indices 1, 2, and i of particular points of the Cartesian system of coordinates are on the line, i.e., not subscripts.) Then eye E1 and eye E2 focus to V1 and V2, respectively, and the binocular image is perceived at C. The situation is similar to that of the random-dot stereograms[18] inducing three-dimensional impressions viewed binocularly: Then the eyes focus on two corresponding dots on a screen and perceive a binocular image behind or in front of the screen if the directions of view cross behind or in front of the screen, respectively. When rays emitted from O along a cone with a vertical axis through O are considered (Fig. 5), $e1$ and $e2$ cross at V = V1 = V2. Then both eyes focus to V where the binocular image also is perceived. All other rays from O do not intersect after refraction. This is the situation if the observer keeps the head oblique with respect to the water surface when $e1$ and $e2$ do not cross (Fig. 6). From this Horváth and Varjú[13] concluded that there is no binocular image formation for obliquely oriented eyes.

However, this is not always true. According to Fig. 6, looking at O with obliquely oriented eyes and focusing with E1 to V1 and with E2 to V2, the observer sees two distinct images V1′ and V2′ (not shown in Fig. 6) somewhere along the noncrossing lines $e1$ and $e2$ if the optical axes of the eyes do not coincide with $e1$ and $e2$. There are two points, K1 and K2, which are the nearest loci from each other along $e1$ and $e2$, respectively, as shown in the enlarged diagram in Fig. 6. The minimum distance between $e1$ and $e2$ is K1 K2. Point K is placed halfway between K1 and K2. Lines $e1$ and $e2$ converge in the plane passing through K and the optical centers of eyes

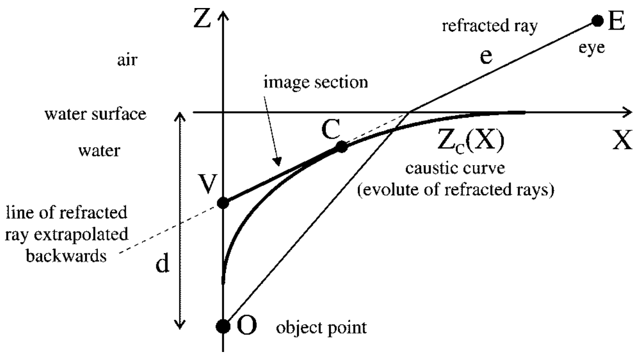

Fig. 1. Geometry of refraction of a ray of light starting from an underwater object point O and entering an aerial eye E. According to different authors,[3–13] C and V are the two possible apparent image points of O.

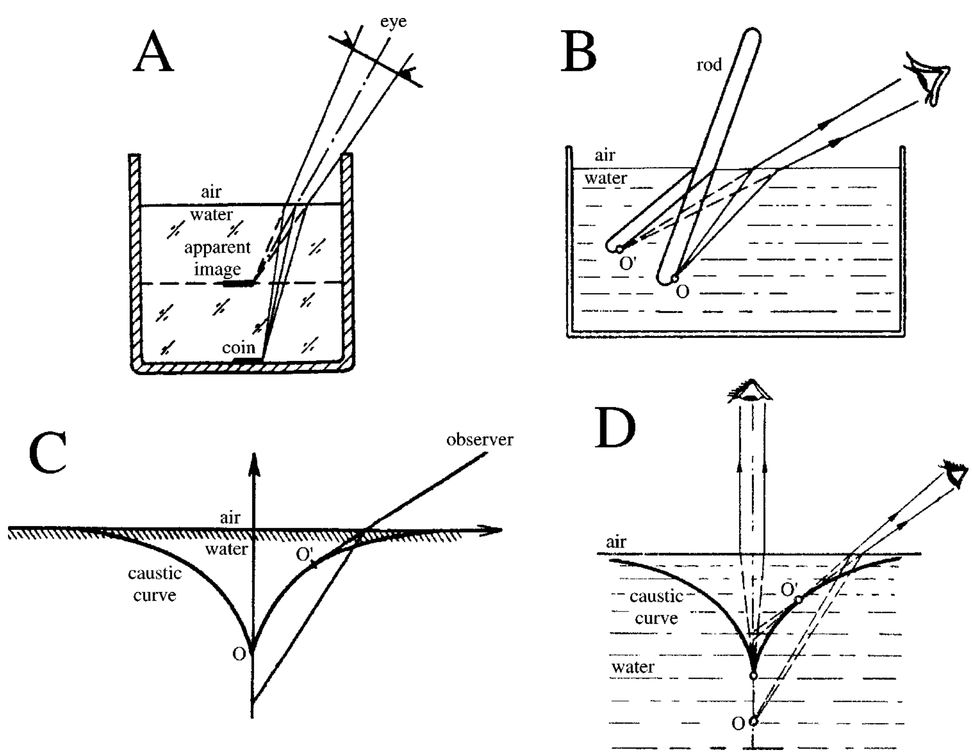

Fig. 2. Representations of the apparent images of underwater objects viewed from air cited from different textbooks. The figures are slightly modified and redrawn after A, Ref. 8, Fig. 2.12; B, Ref. 7, Fig. 248.3; C, Ref. 6, Fig. 28; D, Ref. 7, Fig. 255.8. O: position of an underwater object, O′: apparent position of O.

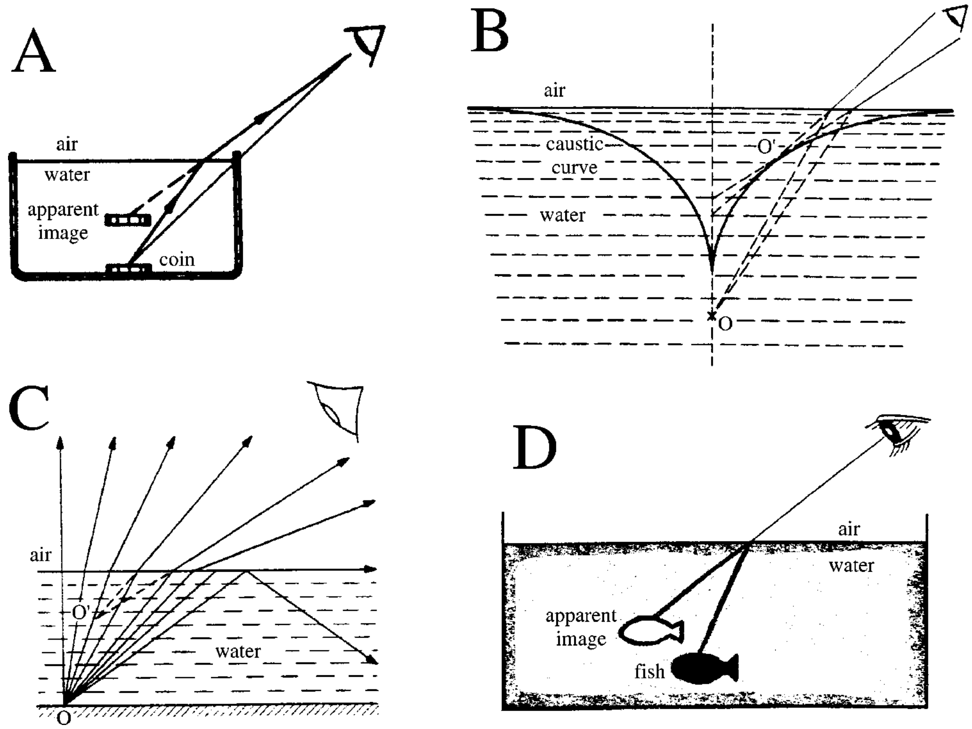

Fig. 3. As Fig. 2 with figures after A, Ref. 12, Fig. 1.36A; B, Ref. 12, Fig. 1.74; C, Ref. 10, Fig. 14.13; D, Ref. 11, Fig. 8.11.

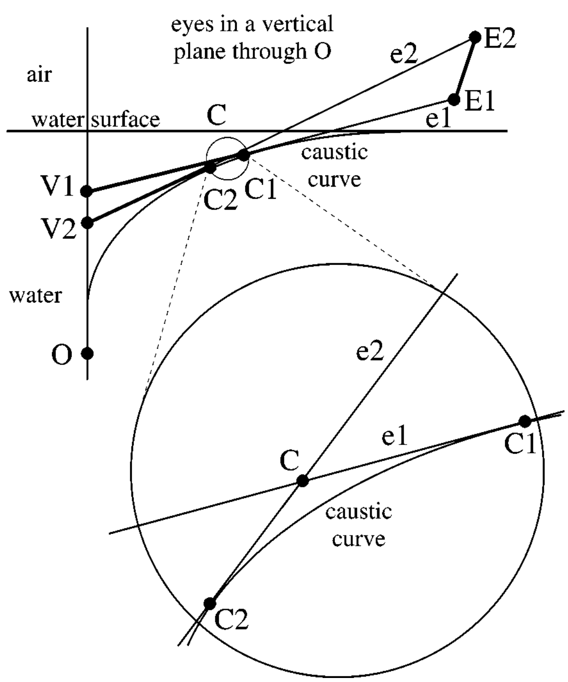

Fig. 4. If eyes E1 and E2 of an aerial observer and the underwater object point O lie in the same vertical plane, lines $e1$ and $e2$ of refracted rays extrapolated backward and entering the eyes intersect at point C. Thus C is the binocular image of O.

E1 and E2, whereas they diverge perpendicularly to this plane. If the viewing directions of both eyes can converge and diverge appropriately in the plane through K and the optical centers of the eyes and perpendicularly to it, respectively, in such a way that the optical axes of the eyes coincide with $e1$ and $e2$, then the mentioned images V1′ and V2′ are fused into a binocular image positioned at K. In other words, binocular image fusion is performed at K by appropriate vergent eye movements if V1′ and V2′ are brought onto the fovea of eye E1 and eye E2, respectively. In humans, the optical axes passing through the fovea and the optical center of the eyes can converge strongly in a plane through the optical centers but can diverge only slightly (a few degrees) perpendicularly to this plane.[15] Thus if the minimum distance K1 K2 is too large, binocular fusion cannot be performed, and the observer sees two distinct images along $e1$ and $e2$ at an indefinite distance. Hence the chance of the existence of the binocular image point K of O is greater, the smaller K1 K2 is. When eye E2 rotates around eye E1 in such a way that the baseline between them changes from vertical to horizontal, then K1, K2 move respectively from C1, C2 to V1, V2 along the image sections, and K moves from C (Fig. 4) to V = V1 = V2 (Fig. 5) as seen in Fig. 6.

The aim of this work is threefold: (1) We calculate the position of the binocular image point K for an underwater object point O viewed by two arbitrarily positioned aerial eyes. (2) Assuming that binocular image fusion is performed by appropriate vergent eye movements, the structure of the underwater binocular visual field determined by the binocular image points K is computed and visualized in different ways as a function of the direction of the baseline between the eyes of an aerial observer. The minimum distance K1 K2 is also calculated as a function of the direction of view and the obliqueness of head tilting. We show that the structure of aerial observer's underwater binocular visual field distorted by refraction is more complex than has been thought previously. With this understanding, we improve the theory of aerial binocular imaging of underwater objects developed earlier by Matthiessen,[3] Kinsler,[4] Kedves,[5] Buchholz,[9] and Horváth and Varjú[13] for the above-mentioned two extreme arrangements of eyes. (3) Finally, some widespread erroneous or incomplete representations of the aerial imaging of underwater objects occurring in different textbooks are analyzed and corrected.

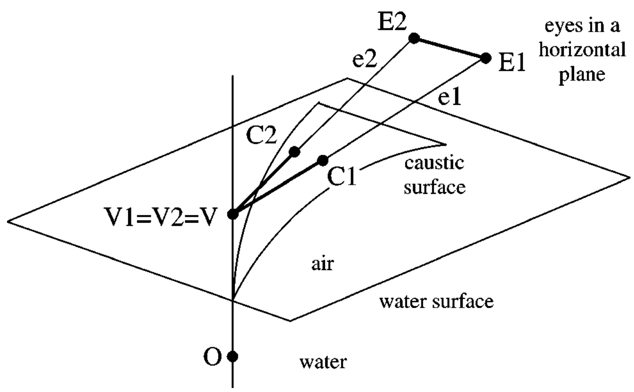

Fig. 5. When the two aerial eyes E1 and E2 lie in a horizontal plane, the refracted rays $e1$ and $e2$ extrapolated backward and entering the eyes intersect at point V. Thus V is the binocular image of O.

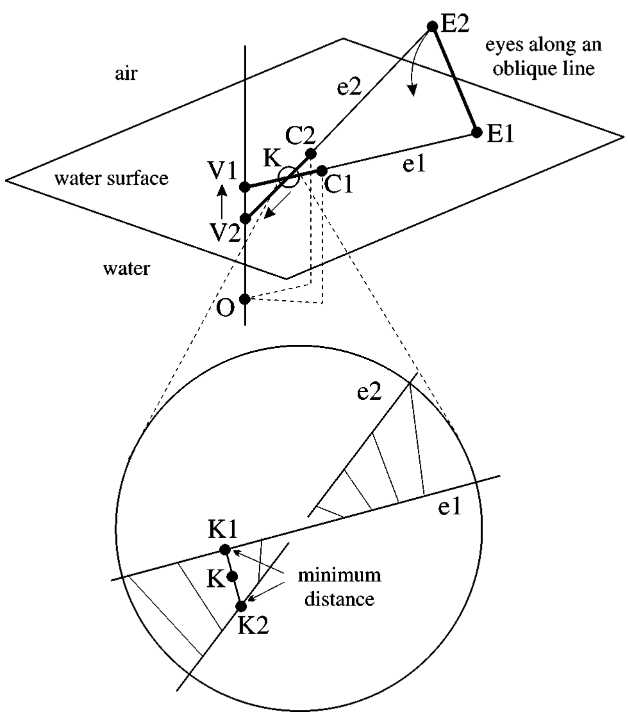

Fig. 6. When the two aerial eyes E1 and E2 lie along an oblique line relative to the water surface, the lines $e1$ and $e2$ of refracted rays extrapolated backward and entering the eyes do not intersect; they avoid each other in space. If the optical axes of the eyes coincide with $e1$ and $e2$ owing to appropriate vergent eye movements, the binocular image of O is K, which halves the minimum distance K1 K2 between the two avoiding lines $e1$ and $e2$.

2. MATERIALS AND METHODS

A. Caustic of Refracted Rays of Light Extrapolated Backward

Consider an underwater object point O, from which rays of light start and propagate in different directions toward the flat water surface, where they are refracted. The so-called caustic surface is the evolute of refracted rays extrapolated backward. This is a cylindrically symmetric surface with a trumpetlike shape, the rotation axis of which is the vertical line passing through O. Its vertical main cross section is the caustic curve $Z_c(X)$, the expression of which is (see Fig. 1)[13]

$$Z_c(X) = -(d/n)\{1 - (n^2 - 1)[-X/(dn^2 - d)]^{2/3}\}^{3/2}, \tag{1}$$

where $n = 1.33$ is the index of refraction of water and d is the depth of O from the water surface. The line of a refracted ray extrapolated backward has two distinguished points, C and V, at which it touches the caustic curve and intersects the vertical line through O, respectively. The straight line between C and V is called the image section (Fig. 1).

B. Position of the Binocular Image Point of an Underwater Object Point for Arbitrary Positions of the Eyes

The distance U between the two aerial eyes E1 and E2 is constant and set as unit ($U = 1$). E1 is fixed to axis Z at a height h from the water surface, while the position of E2 varies on the surface of a sphere, the radius of which is $U = 1$. This sphere is called the unity sphere of possible positions of E2 further on in this paper. The direction of section U connecting the eyes is characterized by angle θ measured from axis Z and by angle φ measured from axis X. Our calculations are restricted to the positions of E2 on the region of the surface of the unity sphere characterized by $0° \leq \theta \leq 180°$ and $0° \leq \varphi \leq 90°$. The underwater binocular visual field for positions of E2 outside this region can be obtained by appropriate mirroring or rotating the corresponding pattern calculated for a given position of E2 within the mentioned region.

Consider a Cartesian system of coordinates, in which the two aerial eyes $\mathrm{E1}(X_{\mathrm{E1}}, Y_{\mathrm{E1}}, Z_{\mathrm{E1}})$ and $\mathrm{E2}(X_{\mathrm{E2}}, Y_{\mathrm{E2}}, Z_{\mathrm{E2}})$ look at the underwater object point $\mathrm{O}(X_{\mathrm{O}}, Y_{\mathrm{O}}, Z_{\mathrm{O}})$. Figure 7 shows the path composed of the underwater section ri ($i = 1$ or 2) and the aerial section ei of the ray of light from O through the point of refraction Ri at the air–water interface toward the eye Ei in three dimensions, where indices $i = 1, 2$ refer to the parameters and symbols belonging to E1 and E2. Figure 8 represents the same path in the vertical plane of points O, Ri, and Ei. Angle α_i of the aerial section ei and angle β_i of the underwater section ri are measured from the water surface. The horizontal distances between Ei and O; Ei and Ri; Ri and O are d_i, a_i, b_i, respectively. From Fig. 8 we can read

$$d_i = [(X_{\mathrm{E}i} - u)^2 + (Y_{\mathrm{E}i} - v)^2]^{1/2}, \qquad i = 1, 2 \tag{2}$$

$$d_i = a_i + b_i, \tag{3}$$

$$a_i = Z_{\mathrm{E}i} \operatorname{cotan} \alpha_i, \tag{4}$$

$$b_i = -w \operatorname{cotan} \beta_i, \tag{5}$$

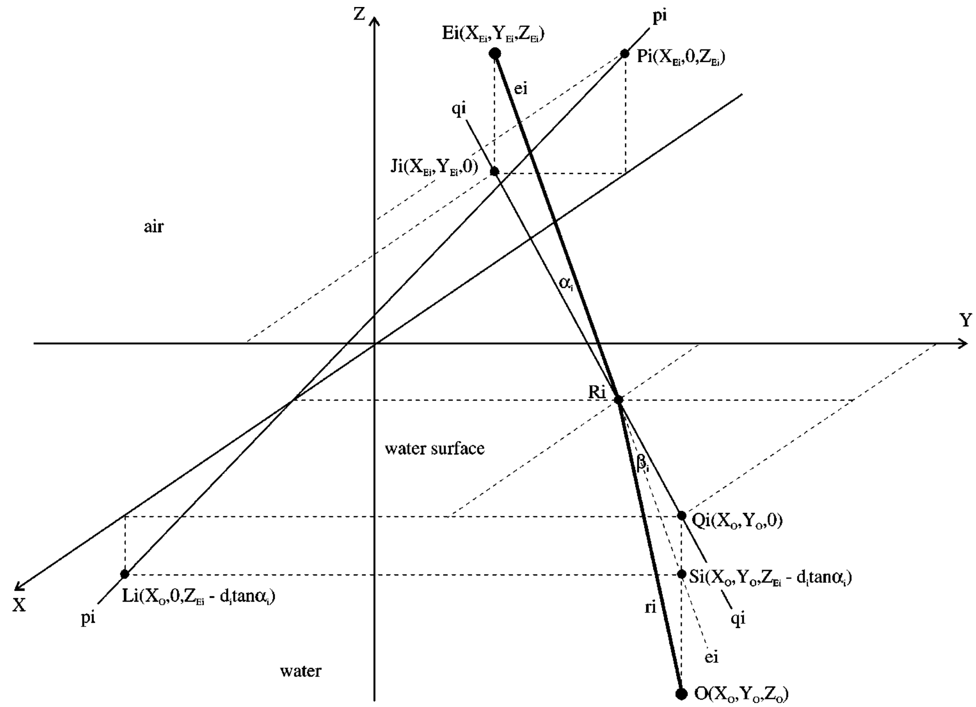

Fig. 7. Underwater (*ri*) and aerial (*ei*) path of a ray of light from an underwater object point O through the point of refraction R*i* at the water surface to the aerial eye E*i*, represented in the system of *XYZ* coordinates, where index $i = 1$ or 2.

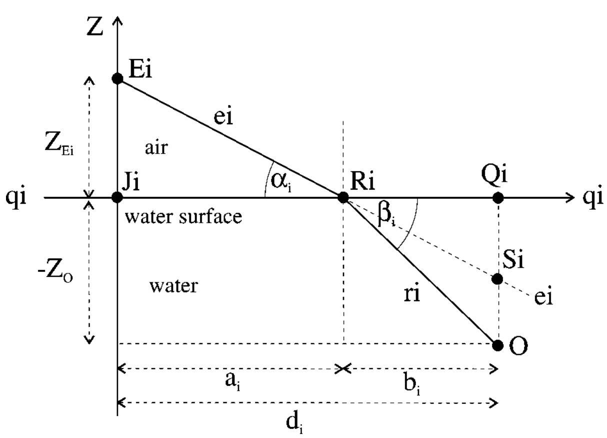

Fig. 8. Path of a ray of light from an underwater object point O through the point of refraction R*i* to the aerial eye E*i* ($i = 1$ or 2) in the vertical plane of refraction.

where u, v, and w are free variables. According to the Snell–Descartes law of refraction,

$$\cos\alpha_i / \cos\beta_i = n = 1.33. \tag{6}$$

Using the trigonometric expressions

$$\operatorname{cotan}\alpha_i = \cos\alpha_i(1 - \cos^2\alpha_i)^{-1/2},$$

$$\operatorname{cotan}\beta_i = \cos\beta_i(1 - \cos^2\beta_i)^{-1/2}, \tag{7}$$

we obtain the following equation of fourth order for the variable $t = \cos^2\alpha_i$:

$$a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 = 0, \quad a_0 = d_i^{\,4} n^4,$$

$$a_1 = -2 d_i^{\,2} n^2 (Z_{\mathrm{E}i}^{\,2} n^2 + w^2 + d_i^{\,2} + d_i^{\,2} n^2),$$

$$a_2 = (Z_{\mathrm{E}i}^{\,2} n^2 + w^2 + d_i^{\,2} + d_i^{\,2} n^2)^2 + 2 d_i^{\,2} n^2 (Z_{\mathrm{E}i}^{\,2} + w^2 + d_i^{\,2}) + 4 Z_{\mathrm{E}i}^{\,2} w^2 n^2,$$

$$a_3 = -2(Z_{\mathrm{E}i}^{\,2} n^2 + w^2 + d_i^{\,2} + d_i^{\,2} n^2)(Z_{\mathrm{E}i}^{\,2} + w^2 + d_i^{\,2}) - 4 Z_{\mathrm{E}i}^{\,2} w^2 (n^2 + 1),$$

$$a_4 = (Z_{\mathrm{E}i}^{\,2} + w^2 + d_i^{\,2})^2 + 4 Z_{\mathrm{E}i}^{\,2} w^2. \tag{8}$$

Solving Eqs. (8) numerically with the use of the tangent method of Newton combined with bisection,[19] we obtain α_i for given $X_{\mathrm{E1}}, Y_{\mathrm{E1}}, Z_{\mathrm{E1}}, X_{\mathrm{E2}}, Y_{\mathrm{E2}}, Z_{\mathrm{E2}}, X_{\mathrm{O}}, Y_{\mathrm{O}}, Z_{\mathrm{O}}$. The line of ray *ei* entering E*i* is described by two equations: (1) The first is the equation of line *pi* passing through P*i* and L*i* positioned in the plane of axes *X* and *Z* in Fig. 7, where *pi* is the projection of line *ei* onto the *X–Z* plane. (2) The second is the equation of line *qi* passing through J*i* and Q*i* positioned in the plane of axes *X* and *Y* in Fig. 7, where *qi* is the projection of line *ei* onto the *X–Y* plane. On the basis of Fig. 7 the Cartesian coordinates of these points are

$$\mathrm{P}i: \ (X_{\mathrm{E}i}, 0, Z_{\mathrm{E}i}), \qquad \mathrm{L}i: \ (X_{\mathrm{O}}, 0, Z_{\mathrm{E}i}\ d_i \tan\alpha_i),$$

$$\mathrm{J}i: \ (X_{\mathrm{E}i}, Y_{\mathrm{E}i}, 0), \qquad \mathrm{Q}i: \ (X_{\mathrm{O}}, Y_{\mathrm{O}}, 0). \tag{9}$$

The equations of the mentioned two projections ($p1, q1$) of line $e1$ are

$$Y_1(X_1) = AX_1 + B, \quad A = (Y_{E1} - Y_O)/(X_{E1} - X_O),$$
$$B = (X_{E1}Y_O - Y_{E1}X_O)/(X_{E1} - X_O),$$
$$Z_1(X_1) = CX_1 + D, \quad C = (d_1 \tan \alpha_1)/(X_{E1} - X_O),$$
$$D = [X_{E1}(Z_{E1} - d_1 \tan \alpha_1) - X_O Z_{E1}]/(X_{E1} - X_O). \tag{10}$$

The equations of the mentioned two projections ($p2, q2$) on line $e2$ are

$$Y_2(X_2) = EX_2 + F, \quad E = (Y_{E2} - Y_O)/(X_{E2} - X_O),$$
$$F = (X_{E2}Y_O - Y_{E2}X_O)/(X_{E2} - X_O),$$
$$Z_2(X_2) = GX_2 + H, \quad G = (d_2 \tan \alpha_2)/(X_{E2} - X_O),$$
$$H = [X_{E2}(Z_{E2} - d_2 \tan \alpha_2) - X_O Z_{E2}]/(X_{E2} - X_O). \tag{11}$$

The distance between two arbitrary points of lines $e1$ and $e2$ is

$$\begin{aligned} d(X_1, X_2) &= \{(X_1 - X_2)^2 + [Y_1(X_1) - Y_2(X_2)]^2 \\ &\quad + [Z_1(X_1) - Z_2(X_2)]^2\}^{1/2} \\ &= \{(X_1 - X_2)^2 + (AX_1 + B - EX_2 - F)^2 \\ &\quad + (CX_1 + D - GX_2 - H)^2\}^{1/2}. \end{aligned} \tag{12}$$

We are looking for the minimum $d^*(X_1^*, X_2^*)$ of distance $d(X_1, X_2)$. Function $d(X_1, X_2)$ has its minimum at the same values of X_1^* and X_2^* where function $d^2(X_1, X_2)$ has; thus the prerequisite of the minimum is

$$\begin{aligned} \partial d^2(X_1^*, X_2^*)/\partial X_1 &= 2(X_1^* - X_2^*) + 2A(AX_1^* + B \\ &\quad - EX_2^* - F) + 2C(CX_1^* + D \\ &\quad - GX_2^* - H) = 0, \\ \partial d^2(X_1^*, X_2^*)/\partial X_2 &= -2(X_1^* - X_2^*) - 2E(AX_1^* + B \\ &\quad - EX_2^* - F) - 2G(CX_1^* + D \\ &\quad - GX_2^* - H) = 0. \end{aligned} \tag{13}$$

Solving the system of Eqs. (13) for X_1^* and X_2^*, we obtain

$$\begin{aligned} &X_1^* = (\gamma\delta + \epsilon\mu)/(\mu^2 - \eta\delta), \quad X_2^* = (\eta X_1^* + \gamma)/\mu, \\ &\eta = 1 + A^2 + C^2, \quad \mu = 1 + AE + CG, \\ &\gamma = A(B - F) + C(D - H), \quad \delta = 1 + E^2 + G^2, \\ &\epsilon = E(F - B) + G(H - D). \end{aligned} \tag{14}$$

Using Eqs. (10), (11), and (14), we find that the Cartesian coordinates of the binocular image point K of O are

$$\begin{aligned} \mathrm{K}: \quad &[(X_1^* + X_2^*)/2, \quad (AX_1^* + B + EX_2^* + F)/2, \\ &(CX_1^* + D + GX_2^* + H)/2]. \end{aligned} \tag{15}$$

Hence the position of K can be calculated as follows: (i) d_1, d_2 are determined from Eq. (2) and α_1, α_2 are calculated from Eqs. (8). (ii) The free variables A, B, C, D, E, F, G, H are determined from Eqs. (10) and (11). (iii) X_1^* and X_2^* are determined from Eqs. (14). (iv) The coordinates of K are calculated from relation (15). The input data are the coordinates X_O, Y_O, Z_O of the underwater object point O and the coordinates X_{E1}, Y_{E1}, Z_{E1}, X_{E2}, Y_{E2}, Z_{E2} of the aerial eyes E1 and E2.

3. RESULTS

Figures 9–11 show how strongly the structure of the underwater world is distorted owing to refraction at the water surface if viewed from air binocularly as a function of the angles φ and θ of eye E2 on the unity sphere. A general feature is that the apparent depth of all underwater points decreases more or less: the greater the horizontal distance of an underwater point from the observer, the smaller its apparent depth. A cubic part of the underwater world is distorted and contracted in a characteristic deep basin-shaped formation. Owing to refraction underwater, straight (i.e., horizontal or vertical) lines are distorted to more- or less-curved lines depending on the direction of view and the position of eye E2. The greatest curvatures of these lines occur approximately along concentric spheres around the observer. Farther away from the observer the curvatures become gradually smaller. At quite large distances from the observer, horizontal or vertical underwater lines approximately keep their horizontal or vertical alignments, but all horizontal lines are apparently seen in the immediate vicinity of the water surface. Figures 9–11 demonstrate also that the structure of the underwater world distorted by refraction depends strongly on the relative positions of the eyes.

Figure 9 shows the binocular image of an underwater vertical-plane quadratic grid for several different positions of eye E2 on the unity sphere. One can see that the binocular image of underwater vertical or horizontal lines suffers only a relatively small apparent distortion if the direction of view is near the vertical, and the image distortions increase if the viewing direction nears the horizontal. Depending on the position of eye E2, the image of horizontal lines is generally a characteristic, approximately mirror-symmetric bell-shaped curve (drawings C, D, F, G, H, I, M, N). For certain eye positions, however, this shape becomes quite asymmetric (drawings E, J, K, L) with large local curvatures and protrusions. The image of vertical lines is usually a typical S-like curve (e.g., drawings C, F, M), but for some positions of eye E2 this

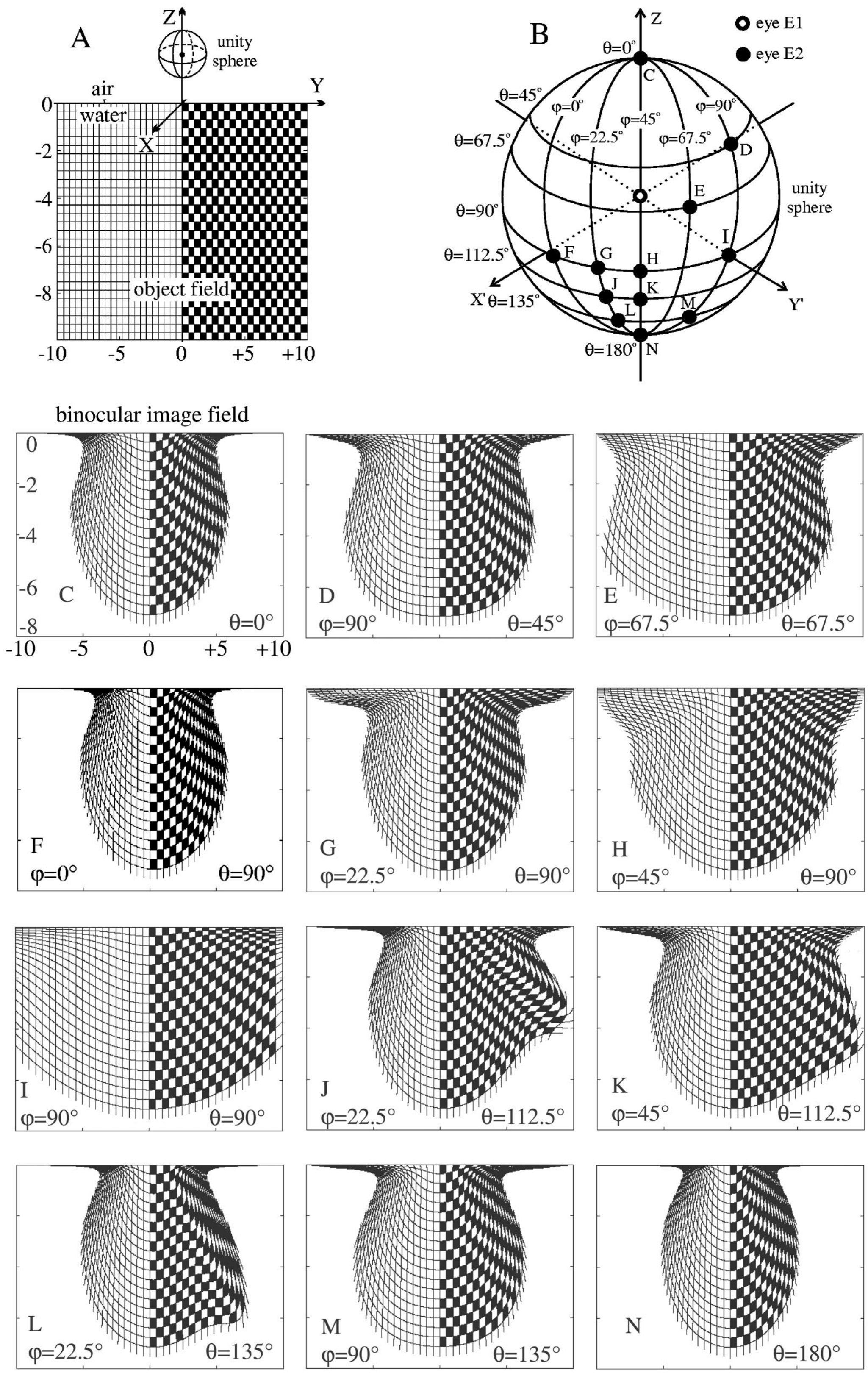

Fig. 9. Binocular imaging of underwater object points in a vertical plane as a function of relative eye positions. A, underwater vertical quadratic grid as object field, consisting of equidistant horizontal and vertical lines, the distance of which is equal to the distance U of the eyes set as unit ($U = 1$). For a better visualization, the cells of the grid are alternately painted white and black on the right half. The coordinates of the fixed aerial eye E1 are $X = 0, Y = 0, Z = 2$. The small circle above the grid represents the unity sphere, at the center of which is E1 and on the surface of which E2 is situated. B, the positions of E2 on the unity sphere for which the binocular images were computed. C–N, binocular image of the underwater grid in A as functions of angles φ and θ of E2 on the unity sphere. In the calculations it was assumed that the binocular image point of every object point is the point K defined in Fig. 6.

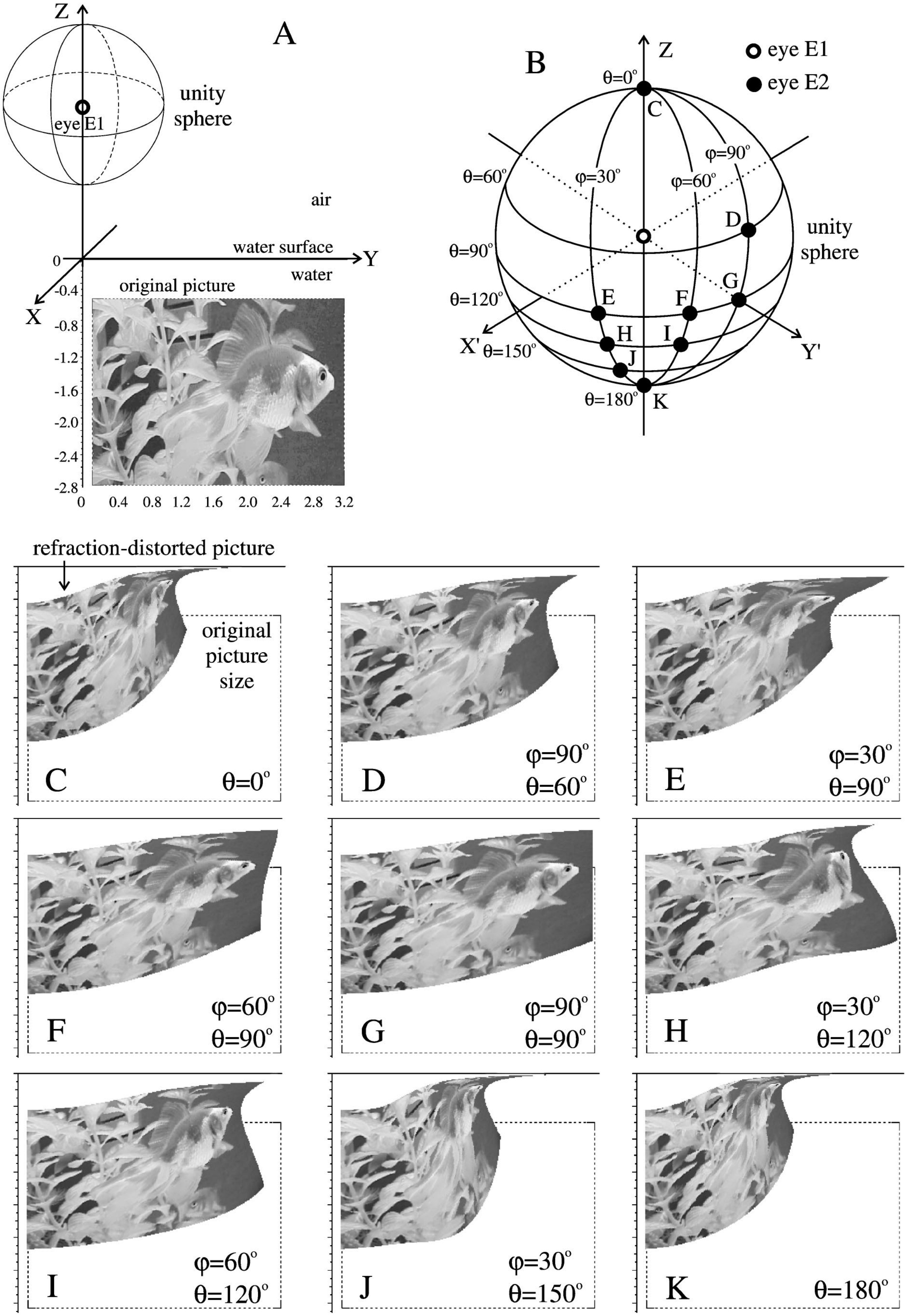

Fig. 10. As Fig. 9, but here the vertical grid is replaced by a picture representing a vertical section of the underwater world in an aquarium with a goldfish and water plants.

shape has a smaller or larger local bulge (e.g., drawings J, K, L). We can see how the square-shaped cells of the underwater world are distorted to elongated or flattened deltoids or rhombi depending on the direction of view and the position of eye E2.

Figure 10 demonstrates how a vertical section of the aquatic world in an aquarium (with a goldfish and water plants) is distorted because of refraction if the observer looks into the water from above with different positions of its eye E2 relative to E1. We see how the shape of the fish, for instance, is distorted, e.g., to a strongly curved S, flattened or elongated depending on the position of eye E2.

The left columns in Figs. 11 and 12 show the binocular image of an underwater horizontal and vertical quadratic plane grid, respectively, as a function of the position of eye E2. The characteristic basin-shaped surfaces in the left column of Fig. 11 demonstrate the apparent shape of the horizontal bottom of lakes that can be seen in everyday life during boating or rowing when the water is transparent, as a function of the position of eye E2. We can see in the left column of Fig. 12 that for certain eye positions (e.g., Figs. 12C–12E) the binocular image of an underwater vertical grid is not a plane but is a slightly curved surface differing more or less from the plane of the underwater vertical object grid. At other positions of eye E2, the binocular image remains exactly two dimensional, the plane of which is parallel to the original vertical grid (Figs. 12B and 12F).

The right columns of Figs. 11 and 12 show the minimum distance K1 K2 between the two avoiding refracted rays of light $e1$ and $e2$ extrapolated backward and entering eyes E1 and E2 as a function of the position of E2. At the half-point of section K1 K2 the binocular image point K of an underwater object point O is formed (Fig. 6), if binocular fusion is performed. The greater this minimum distance K1 K2, the larger vergent eye movements are needed for binocular fusion. In the two special cases of the eye positions studied earlier by Horváth and Varjú[13] and mentioned above, K1 K2 is zero. Then the optical axes of the eyes need only to converge appropriately in a plane through the optical centers of the eyes. However, if K1 K2 differs from zero, an appropriate divergence of the optical axes of the eyes perpendicularly to this plane is also necessary for binocular fusion.

In this paper all calculations are performed for the case in which the height h of the fixed eye E1 above the water surface is $2U$, where $U = 1$ is the eye distance set as unit. With increasing h, the dependence of the refraction-induced apparent distortion of the underwater

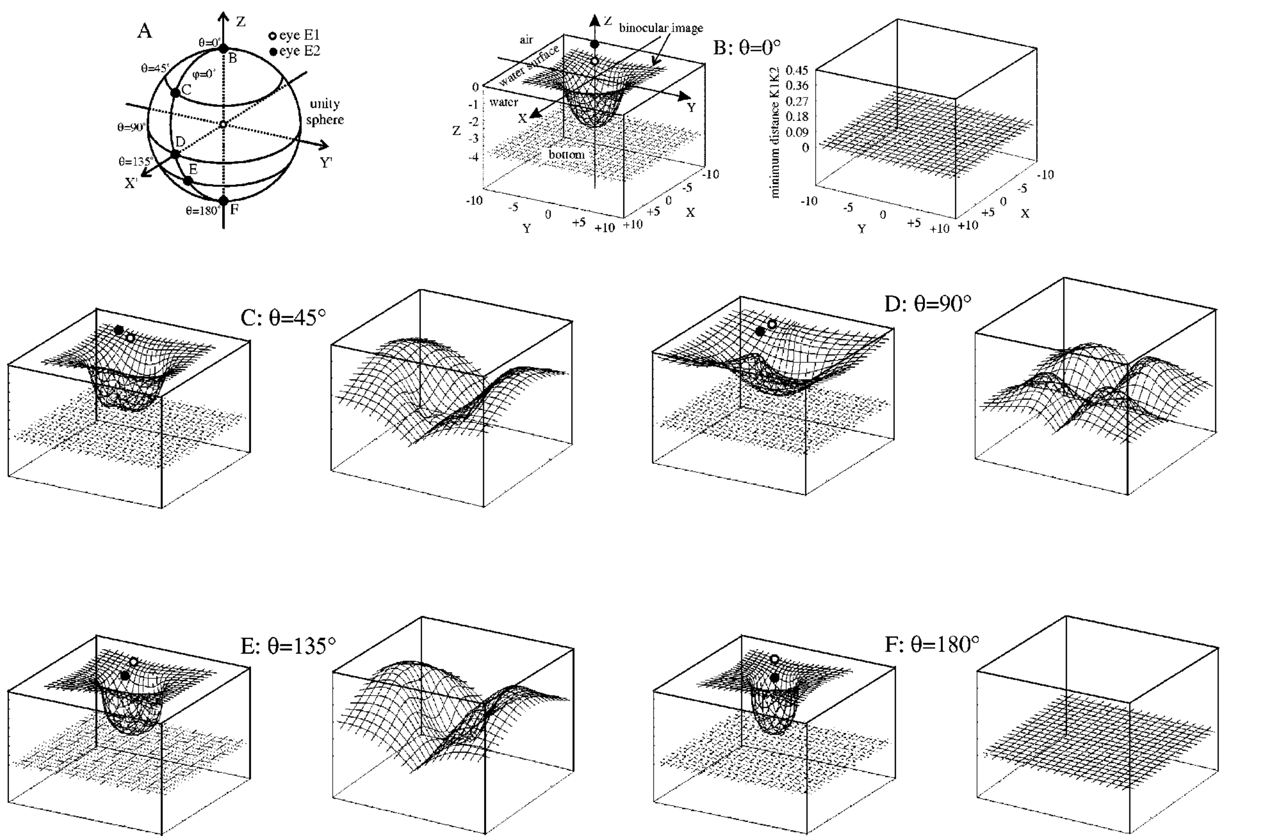

Fig. 11. Binocular imaging of underwater object points in a horizontal plane versus relative eye positions. A: Eye positions for which computations were done. Left column in rows B–F: Binocular image of the horizontal bottom of a shallow (depth $Z = -4$) lake viewed from air through the flat water surface ($Z = 0$) as a function of angle θ of eye E2 with respect to eye E1 for $\varphi = 0°$. The positions of the eyes are shown by dots. In the calculations it was assumed that the binocular image point of every object point is the point K defined in Fig. 6. Right column in rows B–F: The minimum distance K1 K2 between the two nearest points K1 and K2 of lines $e1$ and $e2$ of the refracted rays entering the eyes (Fig. 6) as functions of X and Y in three-dimensional-perspective representation.

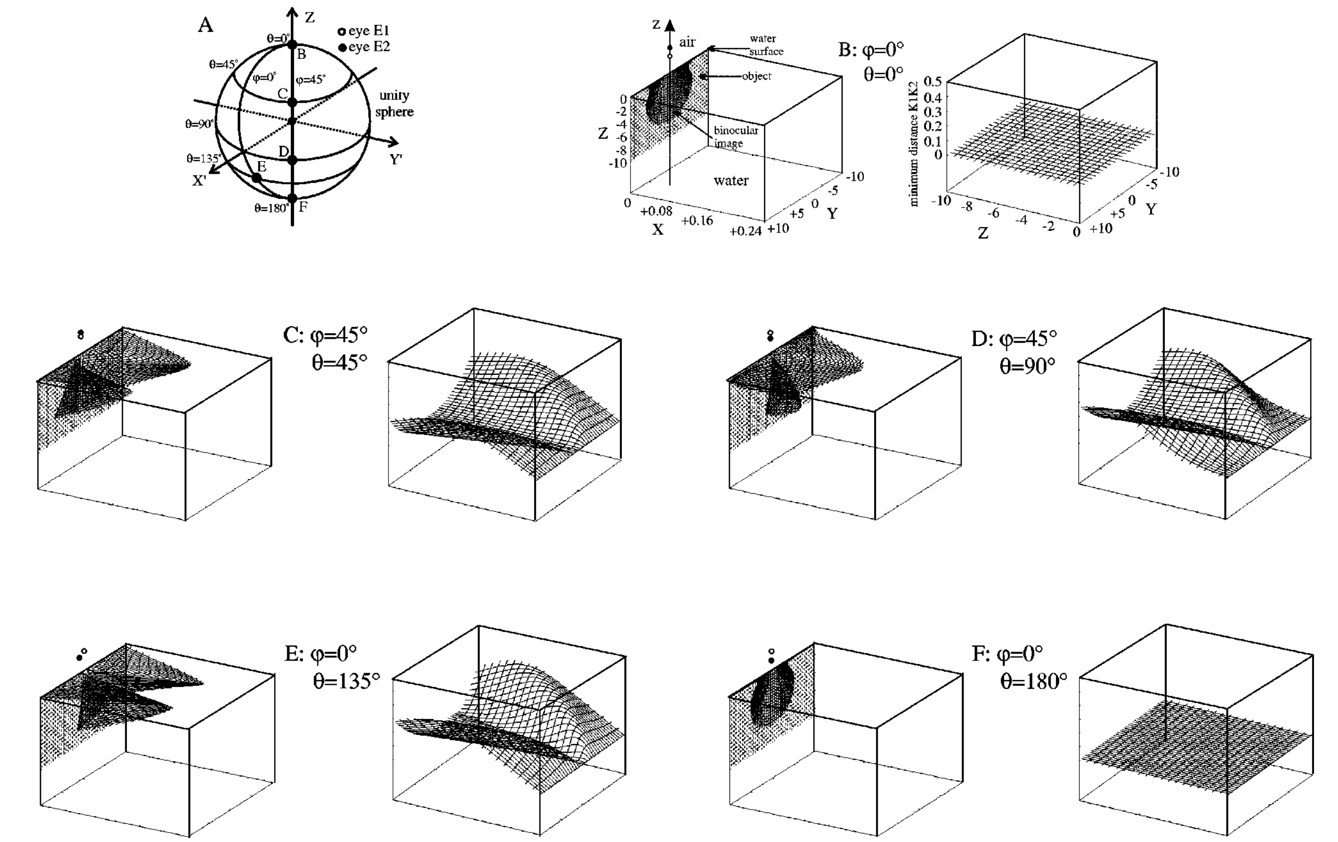

Fig. 12. As Fig. 11, but here the object is an underwater vertical quadratic grid positioned in the plane of axes Y and Z. The grid consists of equidistant horizontal and vertical lines, which are parallel to axes Y and Z; the grid parameter is equal to $U = 1$.

world on the position of eye E2 becomes weaker, but the gross structure of the underwater binocular image field remains qualitatively similar to the patterns in Figs. 9–12.

4. DISCUSSION

We would like to emphasize that in the calculations of Figs. 9–12 it was assumed that the binocular image point of every object point is the point K as defined in Fig. 6. Thus in Figs. 9–12 for every point a vergent eye movement should be performed to fuse the two image points when the eyes are oriented obliquely. In other words, Figs. 9–12 display nothing other than the positions of the binocular points K, since our calculations rest on the assumption that binocular fusion provides the required depth cue in each point of the figures. In reality, only one point in space is fixated, and the remaining image points off this axis produce a retinal disparity. Therefore these figures do not reproduce what we could see when looking into the water from air. Studying the influence of other visual mechanisms, such as retinal disparity or focusing, on the perception of underwater objects was beyond the scope of this paper.

A. Revision and Correction of Some Incomplete or Erroneous Representations of the Aerial Binocular Imaging of Underwater Objects

In the literature, some widespread representations of the aerial imaging of underwater objects are incorrect or at least incomplete. Some examples can be seen in Figs. 2 and 3. Figures 2A-2D and Figs. 3A and 3B show incomplete/incorrect representations reproduced from physics textbooks,[6,8,12] and in Figs. 3C and 3D we show two incomplete/incorrect drawings from textbooks for physicians and biologists.[10,11] In these figures, only one eye or its schematic representation is always shown, and there is no information about the relative position of the second eye. However, without this information, these figures are necessarily incomplete, because the apparent position of the binocular image of an underwater object viewed by two aerial eyes depends strongly on the positions of the eyes, as we have shown in this paper. Therefore these and similar representations of the aerial binocular imaging of underwater objects must be revised and corrected.

In Fig. 2A the apparent image of an underwater coin is correctly shifted toward the water surface, but horizontally it is displaced incorrectly farther away from the eye, which displacement can never occur independently of the relative positions of the eyes. The same error appears in Fig. 2B representing incorrectly the apparent image point O′ of the underwater tip O of a rod submerged into water and viewed from air. In Fig. 2C the apparent image of the underwater object point O viewed from air is O′ positioned along the underwater caustic curve. Similar is the situation in Fig. 2D. However, these figures are correct only if the eyes of the observer are in the same vertical plane through O. Figures 3A and 3B demonstrate that discrepancies can occur in the same textbook. According to Fig. 3A, the apparent image of an underwater coin viewed from air rises vertically, which is correct if the eyes lie along a horizontal line. However, in the same chapter of the book, the apparent image point of the underwater point O is O′ positioned on the underwater caustic curve (Fig. 3D), which is incorrect if the eyes lie along a horizontal line.

The physics textbooks for physicians and biologists generally perpetuate the above-mentioned mistakes or incompletenesses. This is demonstrated in Figs. 3C and 3D. The situation in Fig. 3C is the same as that in Figs. 2C and 2D; the only difference is that in Fig. 3C the underwater caustic curve is not represented. The situation in Fig. 3D is the same as that in Figs. 2A and 2B; in Fig. 3D again the underwater caustic curve is not displayed.

The above brief survey demonstrates well our own experience that the aerial binocular imaging of underwater objects is usually incompletely or erroneously represented in physics textbooks as well as cited and perpetuated without criticism by other books. In 1945 Kinsler[4] wrote about the apparent position of an underwater object when viewed from above the water surface: "... there still exists much confusion and misinformation on the subject. Cases could be cited of textbooks and technical publications of recent date that are either in error or in need of clarification in their treatment of the phenomenon." Unfortunately, these words of Kinsler have not gone out of season. The results and analyses presented here revise and correct these and similar geometric optical incompletenesses and errors.

B. Neglected Dependence of the Underwater Binocular Image Field on the Positions of Aerial Eyes

It is rather surprising that in physics textbooks the dependence of the position of the binocular image of an underwater object on the choice of the paths of refracted rays of light involved in image formation is not taken into account, although this dependency has been emphasized, e.g., by Kinsler,[4] Kedves,[5] and Buchholz.[9] Furthermore, Horváth and Varjú[13] studied the binocular imaging of underwater objects for horizontally and vertically oriented eyes. The reverse optical problem, namely, binocular imaging of aerial objects by underwater eyes positioned along a horizontal or a vertical line, was treated by Horváth and Varjú.[20] From the results of the present work it is obvious that a serious error is made if the incorrect apparent position is used in drawing conclusions as to the true location of underwater objects.

Giving a general theory of the aerial binocular imaging of underwater objects for arbitrary relative positions of the eyes, we improved and extended here the theory of Horváth and Varjú.[13] The complexity of the refraction-induced distortion of the structure of the underwater binocular image field presented in this work (Figs. 9–12) contrasts with the incompleteness or incorrectness of treatment of this classical geometric optical problem in textbooks (Figs. 2 and 3). The major goal of this paper was to clarify this problem and to revise and correct these errors and incompletenesses.

We admit that it would be difficult to test experimentally the computational, geometric optical results presented here. It is not easy to measure the position of the binocular image point of an underwater object point as a function of the relative position of the aerial eyes. Our predictions could be tested by judging the relative size of underwater objects, for example. The experimental test of our theoretical results could be an interesting task for future studies. Nevertheless, some of the results presented here can be tested, at least qualitatively in everyday life. Several features of patterns in Figs. 9–12 can be studied and experienced in an aquarium, in a swimming pool, or during boating in a lake with transparent water if the water surface is flat. The horizontal bottom, vertical walls, underwater objects, and structures and lines on these surfaces and objects can help to investigate qualitatively the refraction-induced apparent distortion of the structure of the underwater world as a function of the relative position of the eyes.

ACKNOWLEDGMENTS

This research was supported by a Humboldt research fellowship from the German Alexander von Humboldt Foundation and by an István Széchenyi scholarship from the Hungarian Ministry of Education to G. Horváth. The manuscript was completed at the Department of Cognitive Neuroscience of the University of Tübingen (Germany) during the Humboldt fellowship of G. Horváth. Many thanks go to Hanspeter Mallot (professor and head of department) for making it possible for G. Horváth and D. Varjú to work quietly in his department. We are grateful to Hansjürgen Dahmen for reading and commenting on an earlier version of the manuscript. We are thankful also for a reviewer's constructive comments.

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